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ON CONFORMALLY FLAT POLYNOMIAL (α,β)-METRICS WITH WEAKLY ISOTROPIC SCALAR CURVATURE

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Abstract. In this paper, we study conformally flat (α,β)-metrics in the form $F=\alpha(1+\sum_{j=1}^{m}a_j(\frac{\beta}{\alpha})^j)$ with $m\geq 2$, where α is a Riemannian metric and β is a 1-form on a smooth manifold M. We prove that if such conformally flat (α,β)-metric F is of weakly isotropic scalar curvature, then it must has zero scalar curvature. Moreover, if $a_{m-1}a_m\neq 0$, then such metric is either locally Minkowskian or Riemannian.

1. Introduction

In Riemannian geometry, the conformal properties of Riemannian metrics have been well studied by many geometers. The study of conformal geometry has played an important position which makes us understanding Riemannian manifolds better. In Finsler geometry, the Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely (see [14, 15]). So the study of conformal properties of a Finsler metric becomes more important and it has been a recent popular trend in Finsler geometry. Two Finsler metrics F and $\widetilde{F}$ on a manifold M are said to be *conformally related* if there is a scalar function $\kappa(x)$ on M such that $F=e^{\kappa(x)}\widetilde{F}$. A Finsler metric which is conformally related to a locally Minkowski metric is called *conformally flat*.

It is one hot issue that how to characterize conformally flat metrics in Finsler geometry. There are many important local and global results in conformal Finsler geometry. In [11], Ichijyo and Hashuiguchi defined a conformally invariant linear connection in a Finsler space with an (α,β)-metric and gave a condition that a Randers metric is conformally flat based on their connection, and they also proved that a Finsler manifold is a conformally Bewarld manifold if and only if it is a Wagner manifold (see [9]). Later, Kikuchi found a conformally invariant Finsler connection and gave a necessary and sufficient

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condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition (see [13]). In [10], Matsumoto, Hojo and Okubo studied conformally Berwald Finsler spaces and its applications to (α, β)-metric by using Kikuchi's conformally invariant Finsler connection. In [12], Kang has proved that any conformally flat Randers metric of scalar flag curvature is projectively flat, moreover, such metrics are completely classified. Further, in [3], Chen and Cheng have proved that a conformally flat weak Einstein polynomial (α, β)-metric with isotropic S-curvature is either a locally Minkowski metric or a Riemann metric. In [4], Chen, He and Shen have proved that conformally flat (α, β)-metrics with constant flag curvature must be trivial (locally Minkowskian or Riemannian).

In Finsler geometry, there are several versions of the definition of scalar curvature. Here we adopt the definition of *scalar curvature* introduced by Akbar-Zadeh ([1]). For a Finsler metric F on an n-dimensional manifold M, denoting **Ric** the *Ricci curvature* of F, the scalar curvature $\mathbf{r}$ of F is defined as

$$\mathbf{r} := g^{ij}\mathbf{Ric}_{ij}, \tag{1.1}$$

where

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{y^i y^j}, \quad (g^{ij}) := (g_{ij})^{-1}$$

and $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$. We say a Finsler metric F to be of *weakly isotropic scalar curvature* if there exist a 1-form $\theta = t_i(x)y^i$ and a scalar function $\mu(x)$ such that

$$\mathbf{r} = n(n-1)[\frac{\theta}{F} + \mu]. \tag{1.2}$$

A Finsler metric F is said be of *isotropic scalar curvature* if

$$\mathbf{r} = n(n-1)\mu(x). \tag{1.3}$$

The above concepts come from the notions of some special Riemannian curvature properties. A Finsler metric F on an n-dimensional manifold M is said to be of *weakly isotropic flag curvature* if its *flag curvature* is a scalar function on TM in the following form:

$$K = \frac{3\eta}{F} + \mu, \tag{1.4}$$

where $\eta = \eta_i(x)y^i$ is a 1-form and $\mu = \mu(x)$ is a scalar function on M. A Finsler metric F is called a *weak Einstein metric* if the Ricci curvature satisfies

$$\mathbf{Ric} = (n-1)(\frac{3\eta}{F} + \mu)F^2. \tag{1.5}$$

One can easily see that (1.4) implies (1.5), and (1.5) implies (1.2) with $\theta = \frac{n+5}{2n}\eta$.

In this paper, we mainly focus on the conformally flat (α, β)-metrics with weakly isotropic scalar curvature and get the following theorem.

Theorem 1.1. *Let $F=\alpha\phi(s), s=\beta/\alpha$ be a conformally flat (α,β)-metric on an n-dimensional manifold M with $\beta\neq 0$ and $n\geq 3$, where $\phi(s)$ is a polynomial of degree m $(m\geq 2)$. If F is of weakly isotropic scalar curvature $\mathbf{r}$, then $\mathbf{r}\equiv 0$.*

According to the above theorem, we obtain the following rigidity result.

Theorem 1.2. *Let $F=\alpha\phi(s), s=\beta/\alpha$ be a conformally flat (α,β)-metric on an n-dimensional manifold M with $n\geq 3$, where $\phi(s)=1+a_1s+a_2s^2+\cdots+a_{m-1}s^{m-1}+a_ms^m$ is a polynomial with $m\geq 2$ and $a_{m-1}a_m\neq 0$. If F is of weakly isotropic scalar curvature, then it must be locally Minkowskian or Riemannian.*

Remark 1.3. For $m=1$, $F=\alpha+\beta$ is Randers metric. In this case, Cheng and Yuan has proved that a conformally flat Randers metric on an n-dimensional manifold M $(n\geq 3)$ with isotropic scalar curvature must be locally Minkowskian or Riemannian (see [7]). For $m\geq 2$, Theorem 1.2 is obviously true for the well-known metric $F=\alpha(1+s)^m$.

We have no idea to remove the condition $a_{m-1}a_m\neq 0$ in Theorem 1.2. For a special case, we have the following result.

Proposition 1.4. *Let $F=\alpha\phi(s)=\alpha(1+a_ms^m)$ be a conformally flat (α,β)-metric on an n-dimensional manifold M with $n\geq 3$ and $m\geq 2$. If F is of weakly isotropic scalar curvature, then it must be locally Minkowskian or Riemannian.*

2. Preliminaries

Let M be an n-dimensional smooth manifold with $n\geq 3$. The points in the tangent bundle TM are denoted by (x,y), where $x\in M$ and $y\in T_xM$. Let $(x^i;y^i)$ be the local coordinates of TM with $y=y^i\frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F:TM\to[0,+\infty)$ such that

(i) F is smooth in $TM\backslash\{0\}$;

(ii) $F(x,\lambda y)=\lambda F(x,y)$ for any $\lambda>0$;

(iii) The fundamental quadratic form

$$g=g_{ik}(x,y)dx^i\otimes dx^k,\quad g_{ik}:=\left[\frac{1}{2}F^2\right]_{y^iy^k} \tag{2.1}$$

is positively definite (see [2]). Here and from now on, the lower index x^i, y^i always means partial derivatives, $F_{y^i}:=\frac{\partial F}{\partial y^i}$, $F_{x^i}:=\frac{\partial F}{\partial x^i}$, $[F^2]_{y^iy^k}:=\frac{\partial^2F^2}{\partial y^i\partial y^k}$, etc.

The spray coefficients are given by

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\} \tag{2.2}$$

which determine the geodesic equation $\ddot{\sigma}^i+2G^i(\sigma,\dot{\sigma})=0$. For any $x\in M$ and $y\in T_xM\backslash\{0\}$, the *Riemann curvature* $R_y:=R^i{}_k(x,y)\frac{\partial}{\partial x^i}\otimes dx^k$ is defined by

$$R^i{}_k=2G^i_{x^k}-G^i_{x^jy^k}y^j+2G^jG^i_{y^jy^k}-G^i_{y^j}G^j_{y^k}, \tag{2.3}$$

the *Ricci curvature* **Ric** is the trace of the Riemann curvature defined by

$$\mathbf{Ric} := R^m{}_m. \tag{2.4}$$

The *Ricci tensor* is

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{y^iy^j}. \tag{2.5}$$

By the homogeneity of **Ric**, we have $\mathbf{Ric} = \mathbf{Ric}_{ij}y^iy^j$. A Finsler metric F is called an *Einstein metric* if

$$\mathbf{Ric} = (n-1)\mu F^2, \tag{2.6}$$

where $\mu = \mu(x)$ is a scalar function on M. A Finsler metric F is called a *weak Einstein metric* if the Ricci curvature satisfies

$$\mathbf{Ric} = (n-1)(\frac{3\eta}{F} + \mu)F^2, \tag{2.7}$$

where $\eta = \eta_i(x)y^i$ is a 1-form.

The *scalar curvature* of F introduced by Akbar-Zadeh is defined by

$$\mathbf{r} := g^{ij}\mathbf{Ric}_{ij}.$$

The scalar curvature of a weak Einstein metric F has the form

$$\mathbf{r} = (n-1)[\frac{(n+5)\eta}{2F} + n\mu] = n(n-1)(\frac{\theta}{F} + \mu), \tag{2.8}$$

which satisfies (1.2). Thus, a weak Einstein metric is of *weakly isotropic scalar curvature.*

An (α,β)-metric is a Finsler metric of the form

$$F = \alpha\phi(s), \qquad s = \beta/\alpha,$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form and $\phi = \phi(s)$ is a positive smooth function. In the following we adopt

$$(a^{ij}) := (a_{ij})^{-1}, \qquad b^i := a^{ij}b_j.$$

It is proved that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi = \phi(s)$ is a positive smooth function on an open interval $(-b_0, b_0)$ satisfying the following condition (see [8]):

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0, \tag{2.9}$$

where $b := \|\beta\|_\alpha = a_{ij}b^ib^j$ is the norm of β respect to α.

Two Finsler metrics F and $\widetilde{F}$ on a manifold M are said to be *conformally related* if there is a scalar function $\kappa(x)$ on M such that $F = e^{\kappa(x)}\widetilde{F}$. Particularly, an (α,β)-metric $F = \alpha\phi(\beta/\alpha)$ is conformally related to a Finsler metric $\widetilde{F}$ if $F = e^{\kappa(x)}\widetilde{F}$ with $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{s}) = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$. In the following, we always use symbols with tilde to denote the corresponding quantities of the metric $\widetilde{F}$. Note that $\alpha = e^\kappa\widetilde{\alpha}$, $\beta = e^\kappa\widetilde{\beta}$, thus $\widetilde{s} = s$.

A Finsler metric which is conformally related to a locally Minkowski metric is said to be *conformally flat*. Thus, an conformally flat (α,β)-metric F has the form $F=e^{\kappa(x)}\widetilde{F}$ where $\widetilde{F}=\widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is a locally Minkowski metric.

Denoting

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\ s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i{}_j:=a^{im}s_{mj},\ s_i:=b^js_{ji},$$

where $b_{i|j}$ denotes the covariant derivative of β with respect to α. Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. The following lemma is well-known.

Lemma 2.1 ([8]). *The geodesic coefficients G^i of $F=\alpha\phi(\beta/\alpha)$ are related to G^i_α by*

$$G^i=G^i_\alpha+\alpha Qs^i{}_0+\{-2Q\alpha s_0+r_{00}\}\{\Psi b^i+\Theta\alpha^{-1}y^i\},$$

where

$$Q:=\frac{\phi'}{\phi-s\phi'},$$
$$\Theta:=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi[(\phi-s\phi')+(B-s^2)\phi'']},$$
$$\Psi:=\frac{\phi''}{2[(\phi-s\phi')+(B-s^2)\phi'']}$$

and $B:=b^2$, $s^i{}_0:=s^i{}_jy^j$, $s_0:=s_iy^i$, $r_{00}:=r_{ij}y^iy^j$, etc.

3. Proof of Theorem 1.1

In this section we will use the skills in Cheng, Shen and Tian's paper [6] to prove Theorem 1.1. Firstly we need to compute the Ricci curvature of F. Assume that $F=\alpha\phi(\beta/\alpha)$ is conformally related to a Finsler metric $\widetilde{F}=\widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ on M. Write $\widetilde{\alpha}=\sqrt{\widetilde{a}_{ij}(x)y^iy^j}$, $\widetilde{\beta}=\widetilde{b}_i(x)y^i$, then

$$a_{ij}=e^{2\kappa}\widetilde{a}_{ij},\quad b_i=e^{\kappa}\widetilde{b}_i,\quad \widetilde{b}:=\|\widetilde{\beta}\|_{\widetilde{\alpha}}=\widetilde{a}_{ij}\widetilde{b}^i\widetilde{b}^j=b.$$

Further, we have

$$b_{j|k}=e^{\kappa(x)}(\widetilde{b}_{j\|k}-\widetilde{b}_k\kappa_j+\widetilde{b}_m\kappa^m\widetilde{a}_{jk}),$$
$${}^{\alpha}\Gamma^m_{ij}={}^{\widetilde{\alpha}}\widetilde{\Gamma}^m_{ij}+\kappa_j\delta^m_i+\kappa_i\delta^m_j-\kappa^m\widetilde{a}_{ij},$$
$$r_{ij}=e^{\kappa(x)}\widetilde{r}_{ij}+\frac{1}{2}e^{\kappa(x)}(-\widetilde{b}_j\kappa_i-\widetilde{b}_i\kappa_j+2\widetilde{b}_m\kappa^m\widetilde{a}_{ij}),$$
$$s_{ij}=e^{\kappa(x)}\widetilde{s}_{ij}+\frac{1}{2}e^{\kappa(x)}(\widetilde{b}_i\kappa_j-\widetilde{b}_j\kappa_i),$$
$$r_i=\widetilde{r}_i+\frac{1}{2}(\widetilde{b}_m\kappa^m\widetilde{b}_i-b^2\kappa_i),\quad r=e^{-\kappa(x)}\widetilde{r},$$
$$s_i=\widetilde{s}_i+\frac{1}{2}(b^2\kappa_i-\widetilde{b}_m\kappa^m\widetilde{b}_i),$$

$$r^m{}_m = e^{-\kappa(x)}\widetilde{r^m}_m + e^{-\kappa(x)}(n-1)\widetilde{b}_m\kappa^m,$$

$$s^m{}_i = e^{-\kappa(x)}\widetilde{s^m}_i + \frac{1}{2}e^{-\kappa(x)}(\widetilde{b}^m\kappa_i - \widetilde{b}_i\kappa^m).$$

Here $\widetilde{b}_{j\|k}$ denote the covariant derivatives of $\widetilde{b}_j$ with respect to $\widetilde{\alpha}$. In the following we adopt $\kappa_i = \frac{\partial\kappa}{\partial x^i}$, $\kappa_{ij} := \frac{\partial^2\kappa}{\partial x^i\partial x^j}$, $\kappa^i := \widetilde{a}^{ij}\kappa_j$, $\widetilde{b}^i := \widetilde{a}^{ij}\widetilde{b}_j$, $f := \widetilde{b}_i\kappa^i$, $f_1 := \kappa_{ij}\widetilde{b}^i y^j$, $f_2 := \kappa_{ij}\widetilde{b}^i\widetilde{b}^j$, $\kappa_0 := \kappa_i y^i$, $\kappa_{00} := \kappa_{ij}y^iy^j$, $\|\nabla\kappa\|^2 := \widetilde{a}^{ij}\kappa_i\kappa_j$.

In order to compute the Ricci curvature of F, we need the following lemma.

Lemma 3.1 ([6]). *For an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the Ricci curvature of F is related to the Ricci curvature ${}^{\alpha}\mathbf{Ric}$ of α by*

$$\mathbf{Ric} = {}^{\alpha}\mathbf{Ric} + RT^m_m, \tag{3.1}$$

where

$$\begin{aligned}
RT^m_m = {} & \frac{{r_{00}}^2}{\alpha^2}\Big[(n-1)c_1 + c_2\Big] + \frac{1}{\alpha}\Big\{r_{00}s_0[(n-1)c_3 + c_4] + r_{00}r_0[(n-1)c_5 + c_6] \\
& + r_{00|0}[(n-1)c_7 + c_8]\Big\} + \Big\{{s_0}^2[(n-1)c_9 + c_{10}] + (rr_{00} - r_0^2)c_{11} \\
& + r_0s_0[(n-1)c_{12} + c_{13}] + (r_{00}r^m{}_m - r_{0m}r^m{}_0 + r_{00|m}b^m - r_{0m|0}b^m)c_{14} \\
& + r_{0m}s^m{}_0[(n-1)c_{15} + c_{16}] + s_{0|0}[(n-1)c_{17} + c_{18}] + s_{0m}s^m{}_0c_{19}\Big\} \\
& + \alpha\Big\{rs_0c_{20} + s_ms^m{}_0[(n-1)c_{21} + c_{22}] + (3s_mr^m{}_0 - 2s_0r^m{}_m + 2r_ms^m{}_0 \\
& - 2s_{0|m}b^m + s_{m|0}b^m)c_{23} + s^m{}_{0|m}c_{24}\Big\} \\
& + \alpha^2\Big\{s_ms^mc_{25} + s^i{}_ms^m{}_ic_{26}\Big\}.
\end{aligned}$$

Here the c_i, $i = 1, 2, \dots, 26$ are the functions only in s and more details can be found in [6].

By the assumption of Theorem 1.1, $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is locally Minkowskian, thus the metric $\widetilde{\alpha}$ is flat and $\widetilde{b}_{j\|k} = 0$ (see [5]). Further, it is obvious that $\widetilde{b} = b = constant \neq 0$. By $\alpha = e^{\kappa}\widetilde{\alpha}$, we have

$${}^{\alpha}\mathbf{Ric} = -(n-2)\kappa_{00} + (n-2)\kappa_0^2 - \widetilde{a}^{ij}\kappa_{ij}\widetilde{\alpha}^2 - (n-2)\|\nabla\kappa\|^2\widetilde{\alpha}^2. \tag{3.2}$$

By Lemma 3.1, we can rewrite the Ricci curvature of F by

$$\mathbf{Ric} = -(n-2)\kappa_{00} + (n-2)\kappa_0^2 - \widetilde{a}^{ij}\kappa_{ij}\widetilde{\alpha}^2 - (n-2)\|\nabla\kappa\|^2\widetilde{\alpha}^2 + RT^m_m. \tag{3.3}$$

In order to compute (3.3) we need the following formulae.

$$\begin{aligned}
&{r_{00}}^2/\alpha^2 = f^2\widetilde{\alpha}^2 - 2fs\kappa_0\widetilde{\alpha} + s^2\kappa_0^2, \\
&r_{00}s_0/\alpha = \frac{1}{2}(b^2f\kappa_0\widetilde{\alpha} - f^2s\widetilde{\alpha}^2 - b^2s\kappa_0^2 + fs^2\kappa_0\widetilde{\alpha}), \\
&r_{00}r_0/\alpha = -\frac{1}{2}(b^2f\kappa_0\widetilde{\alpha} - f^2s\widetilde{\alpha}^2 - b^2s\kappa_0^2 + fs^2\kappa_0\widetilde{\alpha}), \\
&r_{00|0}/\alpha = (f_1 - 2f\kappa_0)\widetilde{\alpha} + (3\kappa_0^2 - \kappa_{00})s - \|\nabla\kappa\|^2s\widetilde{\alpha}^2,
\end{aligned}$$

$$s_0{}^2 = \frac{1}{4}(b^2\kappa_0 - fs\widetilde{\alpha})^2, \quad rr_{00} - r_0^2 = -\frac{1}{4}(b^2\kappa_0 - fs\widetilde{\alpha})^2,$$
$$r_0 s_0 = -\frac{1}{4}(b^2\kappa_0 - fs\widetilde{\alpha})^2, \quad r_{00} r^m{}_m = (n-1)f(f\widetilde{\alpha}^2 - s\kappa_0\widetilde{\alpha}),$$
$$r_{0m}r^m{}_0 = \frac{1}{4}(4f^2\widetilde{\alpha}^2 - 6fs\kappa_0\widetilde{\alpha} + \|\nabla\kappa\|^2 s^2\widetilde{\alpha}^2 + \kappa_0^2 b^2),$$
$$r_{00|m}b^m = (f_2 - f^2)\widetilde{\alpha}^2 + fs\kappa_0\widetilde{\alpha} - f_1 s\widetilde{\alpha} - \|\nabla\kappa\|^2 s^2\widetilde{\alpha}^2 + \kappa_0^2 b^2,$$
$$r_{0m|0}b^m = \frac{1}{2}[-(f^2 + \|\nabla\kappa\|^2 b^2 + \|\nabla\kappa\|^2 s^2)\widetilde{\alpha}^2 + (f_1 + f\kappa_0)s\widetilde{\alpha} + (2\kappa_0^2 - \kappa_{00})b^2],$$
$$r_{0m}s^m{}_0 = \frac{1}{4}(\|\nabla\kappa\|^2 s^2\widetilde{\alpha}^2 - \kappa_0^2 b^2),$$
$$s_{0|0} = \frac{1}{2}(\|\nabla\kappa\|^2 b^2 - f^2)\widetilde{\alpha}^2 + \frac{1}{2}(2f\kappa_0 - f_1)s\widetilde{\alpha} - b^2\kappa_0^2 + \frac{1}{2}b^2\kappa_{00},$$
$$s_{0m}s^m{}_0 = \frac{1}{4}(-\|\nabla\kappa\|^2 s^2\widetilde{\alpha}^2 + 2fs\kappa_0\widetilde{\alpha} - \kappa_0^2 b^2), \quad \alpha r s_0 = 0,$$
$$\alpha s_m s^m{}_0 = \frac{1}{4}(f^2 - \|\nabla\kappa\|^2 b^2)s\widetilde{\alpha}^2,$$
$$3\alpha s_m r^m{}_0 = \frac{3}{4}(2f\kappa_0 b^2 - \|\nabla\kappa\|^2 s b^2\widetilde{\alpha} - f^2 s\widetilde{\alpha})\widetilde{\alpha},$$
$$2\alpha s_0 r^m{}_m = (n-1)(b^2 f\kappa_0 - f^2 s\widetilde{\alpha})\widetilde{\alpha}, \quad 2\alpha r_m s^m{}_0 = \frac{1}{2}(\|\nabla\kappa\|^2 b^2 - f^2)s\widetilde{\alpha}^2,$$
$$2\alpha s_{0|m}b^m = (\|\nabla\kappa\|^2 b^2 s\widetilde{\alpha} - f_2 s\widetilde{\alpha} + f_1 b^2 - fb^2\kappa_0)\widetilde{\alpha},$$
$$\alpha s_{m|0}b^m = \frac{1}{2}(\|\nabla\kappa\|^2 b^2 s\widetilde{\alpha} - fb^2\kappa_0)\widetilde{\alpha},$$
$$\alpha s^m{}_{0|m} = \frac{1}{2}[(n-3)f\kappa_0 - (n-3)\|\nabla\kappa\|^2 s\widetilde{\alpha} + f_1 - \widetilde{a}^{ml}\kappa_{ml}s\widetilde{\alpha}]\widetilde{\alpha},$$
$$\alpha^2 s_m s^m = \frac{1}{4}(\|\nabla\kappa\|^2 b^2 - f^2)b^2\widetilde{\alpha}^2, \quad \alpha^2 s^i{}_m s^m{}_i = \frac{1}{2}(f^2 - \|\nabla\kappa\|^2 b^2)\widetilde{\alpha}^2.$$

Substituting the above formulae into RT^m_m, we obtain the following lemma.

Lemma 3.2. *Let $F = e^\kappa\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is locally Minkowskian. Then the Ricci curvature of F is determined by*

$$\mathbf{Ric} = D_1\|\nabla\kappa\|^2\widetilde{\alpha}^2 + D_2\kappa_0^2 + D_3\kappa_0 f\widetilde{\alpha} + D_4 f^2\widetilde{\alpha}^2 + D_5 f_1\widetilde{\alpha} + D_6\widetilde{\alpha}^2 + D_7\kappa_{00}, \tag{3.4}$$

where

$$D_1 := \frac{1}{4}\left[-3c_{14} + c_{16} - c_{19} + c_{15}(n-1)\right]s^2 + \frac{1}{4}[(1-n)c_{21} - c_{22} - 3c_{23}]b^2 s$$
$$+ \frac{1}{2}[(2-2n)c_7 - 2c_8 + (3-n)c_{24}]s + \frac{1}{4}c_{25}b^4$$
$$+ \frac{1}{2}[c_{14} + (n-1)c_{17} + c_{18} - c_{26}]b^2 + 2 - n,$$
$$D_2 := [(n-1)c_1 + c_2]s^2 + \frac{1}{2}[(n-1)(c_5 - c_3) - c_4 + c_6]b^2 s$$

$$
\begin{aligned}
&+3[(n-1)c_7+c_8]s+\frac{1}{4}[(n-1)(c_9-c_{12})+c_{10}-c_{11}-c_{13}]b^4\\
&-\frac{1}{4}[c_{14}+c_{16}+4(n-1)c_{17}+4c_{18}+c_{19}+c_{15}(n-1)]b^2+n-2,\\
D_3:=\ &\frac{1}{2}[(n-1)(c_3-c_5)+c_4-c_6](b^2+s^2)\\
&+\frac{1}{2}[(n-1)(c_{12}-c_9)-c_{10}+c_{11}+c_{13}]b^2s\\
&-\frac{1}{2}[4(n-1)c_1+4c_2+(2n-6)c_{14}-(2n-2)c_{17}-2c_{18}-c_{19}]s\\
&-(n-3)b^2c_{23}-2[(n-1)c_7+c_8]+\frac{1}{2}(n-3)c_{24},\\
D_4:=\ &\frac{1}{4}[(n-1)(c_9-c_{12})+c_{10}-c_{11}-c_{13}]s^2\\
&+\frac{1}{4}[(n-1)(-2c_3+2c_5+c_{21})-2c_4+2c_6+c_{22}+(4n-9)c_{23}]s\\
&-\frac{1}{4}c_{25}b^2+[(n-1)c_1+c_2]\\
&+\frac{1}{2}[(2n-5)c_{14}-(n-1)c_{17}-c_{18}+c_{26}],\\
D_5:=\ &(n-1)c_7+c_8-\frac{3}{2}sc_{14}-\frac{1}{2}[(n-1)c_{17}+c_{18}]-b^2c_{23}+\frac{1}{2}c_{24},\\
D_6:=\ &\overline{D}_6\widetilde{a}^{ij}\kappa_{ij}+\widetilde{D}_6f_2,\\
\overline{D}_6:=\ &-(1+\frac{1}{2}sc_{24})=-\frac{\phi}{(\phi-s\phi')},\\
\widetilde{D}_6:=\ &sc_{23}+c_{14}=\frac{\phi\phi''}{(\phi-s\phi')[(\phi-s\phi')+(b^2-s^2)\phi'']},\\
D_7:=\ &-[(n-1)c_7+c_8]s+\frac{1}{2}[(n-1)c_{17}+c_{14}+c_{18}]b^2+2-n.
\end{aligned}
$$

We can see $D_1, D_2, D_3, D_4, D_5, D_7$ are the functions only in s and are independent of $\widetilde{\alpha}, \kappa_0, \kappa_{00}, f, f_1, f_2, \widetilde{a}^{ij}\kappa_{ij}$.

Now let us give the formula of scalar curvature $\mathbf{r}$.

Lemma 3.3. *Let $F = e^{\kappa}\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is locally Minkowskian. Then the scalar curvature of F is determined by*

$$
\mathbf{r}=\frac{1}{2}e^{-2\kappa}\rho^{-1}\left(\Sigma_1-\left(\tau+\eta\lambda^2\right)\Sigma_2-\frac{\lambda\eta}{\widetilde{\alpha}}\Sigma_3-\frac{\eta}{\widetilde{\alpha}^2}\Sigma_4\right), \tag{3.5}
$$

where the detail of $\Sigma_i, i = 1, 2, 3, 4$ and $\rho, \tau, \eta, \lambda$ are in the proof.

Proof. For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ on a manifold M, the fundamental tensor of F is given by (see [8])

$$
g_{ij}=\rho a_{ij}+\rho_0b_ib_j+\rho_1(b_i\alpha_j+b_j\alpha_i)+\rho_2\alpha_i\alpha_j, \tag{3.6}
$$

where $\alpha_i := \alpha_{y^i}$ and

$$\rho := \phi(\phi - s\phi'), \qquad \rho_0 := \phi\phi'' + \phi'\phi',$$

$$\rho_1 := -s(\phi\phi'' + \phi'\phi') + \phi\phi', \qquad \rho_2 := s\{s(\phi\phi'' + \phi'\phi') - \phi\phi'\}.$$

The inverse(g^{ij}) of (g_{ij}) is given by (see [8])

$$g^{ij} = \rho^{-1}\{a^{ij} - \tau b^i b^j - \eta Y^i Y^j\}, \tag{3.7}$$

where

$$\tau := \frac{\delta}{1+\delta b^2}, \quad \delta := \frac{\rho_0 - \varepsilon^2\rho_2}{\rho}, \quad \varepsilon := \frac{\rho_1}{\rho_2},$$

$$\eta := \frac{\mu}{1+Y^2\mu}, \quad \mu := \frac{\rho_2}{\rho}, \quad Y := \sqrt{A_{ij}Y^iY^j}, \quad A_{ij} := a_{ij} + \delta b_i b_j$$

and

$$Y^i := \frac{y^i}{\alpha} + \lambda b^i, \qquad \lambda := \frac{\varepsilon - \delta s}{1+\delta b^2}.$$

Further, we can rewrite (g^{ij}) by

$$g^{ij} = \rho^{-1}\{a^{ij} - (\tau + \eta\lambda^2)b^i b^j - \frac{\lambda\eta}{\alpha}(b^i y^j + b^j y^i) - \frac{\eta}{\alpha^2}y^i y^j\}, \tag{3.8}$$

and the $(\widetilde{g}^{ij})$ of $\widetilde{F}$ is

$$\begin{aligned}\widetilde{g}^{ij} &= \rho^{-1}\{\widetilde{a}^{ij} - (\tau + \eta\lambda^2)\widetilde{b}^i\widetilde{b}^j - \frac{\lambda\eta}{\widetilde{\alpha}}(\widetilde{b}^i y^j + \widetilde{b}^j y^i) - \frac{\eta}{\widetilde{\alpha}^2}y^i y^j\}\\ &= e^{2\kappa(x)}g^{ij}.\end{aligned} \tag{3.9}$$

By (1.1) we know the scalar curvature $\mathbf{r}$ of F is

$$\mathbf{r} = \frac{1}{2}e^{-2\kappa}\rho^{-1}\{\widetilde{a}^{ij} - (\tau + \eta\lambda^2)\widetilde{b}^i\widetilde{b}^j - \frac{\lambda\eta}{\widetilde{\alpha}}(\widetilde{b}^i y^j + \widetilde{b}^j y^i) - \frac{\eta}{\widetilde{\alpha}^2}y^i y^j\}\mathbf{Ric}_{y^iy^j}. \tag{3.10}$$

In order to compute (3.10), we need the derivatives of **Ric**,

$$\begin{aligned}\mathbf{Ric}_{y^i} = {} & D_{1,s}s_{y^i}\|\nabla\kappa\|^2\widetilde{\alpha}^2 + D_1\|\nabla\kappa\|^2(\widetilde{\alpha}^2)_{y^i} + D_{2,s}s_{y^i}\kappa_0^2 + 2D_2\kappa_i\kappa_0\\ & + D_{3,s}s_{y^i}\kappa_0 f\widetilde{\alpha} + D_3 f(\kappa_i\widetilde{\alpha} + \kappa_0\widetilde{\alpha}_{y^i}) + D_{4,s}s_{y^i}f^2\widetilde{\alpha}^2 + D_4 f^2(\widetilde{\alpha}^2)_{y^i}\\ & + D_{5,s}s_{y^i}f_1\widetilde{\alpha} + D_5(\kappa_{ji}\widetilde{b}^j\widetilde{\alpha} + f_1\widetilde{\alpha}_{y^i}) + D_{6,s}s_{y^i}\widetilde{\alpha}^2 + D_6(\widetilde{\alpha}^2)_{y^i}\\ & + D_{7,s}s_{y^i}\kappa_{00} + D_7(\kappa_{i0} + \kappa_{0i}),\\ \mathbf{Ric}_{y^iy^j} = {} & D_{1,ss}s_{y^i}s_{y^j}\|\nabla\kappa\|^2\widetilde{\alpha}^2 + D_{1,s}s_{y^iy^j}\|\nabla\kappa\|^2\widetilde{\alpha}^2 + D_{1,s}s_{y^i}\|\nabla\kappa\|^2(\widetilde{\alpha}^2)_{y^j}\\ & + D_{1,s}s_{y^j}\|\nabla\kappa\|^2(\widetilde{\alpha}^2)_{y^i} + D_1\|\nabla\kappa\|^2(\widetilde{\alpha}^2)_{y^iy^j} + D_{2,ss}s_{y^i}s_{y^j}\kappa_0^2\\ & + D_{2,s}s_{y^iy^j}\kappa_0^2 + 2D_{2,s}s_{y^i}\kappa_j\kappa_0 + 2D_{2,s}s_{y^j}\kappa_i\kappa_0 + 2D_2\kappa_i\kappa_j\\ & + D_{3,ss}s_{y^i}s_{y^j}\kappa_0 f\widetilde{\alpha} + D_{3,s}s_{y^iy^j}\kappa_0 f\widetilde{\alpha} + D_{3,s}s_{y^i}(f\kappa_j\widetilde{\alpha} + f\kappa_0\widetilde{\alpha}_{y^j})\\ & + D_{3,s}s_{y^j}(f\kappa_i\widetilde{\alpha} + f\kappa_0\widetilde{\alpha}_{y^i}) + D_3 f(\kappa_i\widetilde{\alpha}_{y^j} + \kappa_j\widetilde{\alpha}_{y^i} + \kappa_0\widetilde{\alpha}_{y^iy^j})\\ & + D_{4,ss}s_{y^i}s_{y^j}f^2\widetilde{\alpha}^2 + D_{4,s}s_{y^iy^j}f^2\widetilde{\alpha}^2 + D_{4,s}s_{y^i}f^2(\widetilde{\alpha}^2)_{y^j}\\ & + D_{4,s}s_{y^j}f^2(\widetilde{\alpha}^2)_{y^i} + D_4 f^2(\widetilde{\alpha}^2)_{y^iy^j} + D_{5,ss}s_{y^i}s_{y^j}f_1\widetilde{\alpha}\end{aligned}$$

$$
\begin{aligned}
&+ D_{5,s}s_{y^iy^j}f_1\widetilde{\alpha} + D_{5,s}s_{y^i}(\kappa_{ij}\widetilde{b}^i\widetilde{\alpha} + f_1\widetilde{\alpha}_{y^j})\\
&+ D_{5,s}s_{y^j}(\kappa_{ji}\widetilde{b}^j\widetilde{\alpha} + f_1\widetilde{\alpha}_{y^i}) + D_5(\kappa_{mi}\widetilde{b}^m\widetilde{\alpha}_{y^j} + \kappa_{mj}\widetilde{b}^m\widetilde{\alpha}_{y^i} + f_1\widetilde{\alpha}_{y^iy^j})\\
&+ D_{6,ss}s_{y^i}s_{y^j}\widetilde{\alpha}^2 + D_{6,s}s_{y^iy^j}\widetilde{\alpha}^2 + D_{6,s}s_{y^i}(\widetilde{\alpha}^2)y^j + D_{6,s}s_{y^j}(\widetilde{\alpha}^2)_{y^i}\\
&+ D_6(\widetilde{\alpha}^2)_{y^iy^j} + D_{7,ss}s_{y^i}s_{y^j}\kappa_{00} + D_{7,s}s_{y^iy^j}\kappa_{00} + D_{7,s}s_{y^i}(\kappa_{j0} + \kappa_{0j})\\
&+ D_{7,s}s_{y^j}(\kappa_{i0} + \kappa_{0i}) + 2D_7\kappa_{ij}.
\end{aligned}
$$

Here we adopt $D_{i,s} := \frac{\partial D_i}{\partial s}$, $D_{i,ss} := \frac{\partial^2 D_i}{\partial s^2}$, $i = 1, 2, \ldots, 7$.
Substituting $\mathbf{Ric}_{y^iy^j}$ into (3.10), we obtain (3.5) with Σ_1, Σ_2, Σ_3, Σ_4 as

$$
\begin{aligned}
\Sigma_1 &:= \widetilde{a}^{ij}\mathbf{Ric}_{y^iy^j}\\
&= D_{1,ss}\|\nabla\kappa\|^2(b^2 - s^2) + D_{1,s}\|\nabla\kappa\|^2(1 - n)s + 2nD_1\|\nabla\kappa\|^2\\
&\quad + \frac{1}{\widetilde{\alpha}^2}[D_{2,ss}\kappa_0^2(b^2 - s^2) - (3 + n)D_{2,s}\kappa_0^2 s] + \frac{4}{\widetilde{\alpha}}(D_{2,s}\kappa_0 f) + 2D_2\|\nabla\kappa\|^2\\
&\quad + \frac{1}{\widetilde{\alpha}}[D_{3,ss}\kappa_0 f(b^2 - s^2) - (n + 1)D_{3,s}\kappa_0 fs + (n + 1)D_3\kappa_0 f]\\
&\quad + 2D_{3,s}f^2 + D_{4,ss}f^2(b^2 - s^2) + D_{4,s}f^2(1 - n)s + 2nD_4 f^2\\
&\quad + \frac{1}{\widetilde{\alpha}}[D_{5,ss}f_1(b^2 - s^2) - (n + 1)D_{5,s}f_1 s + (n + 1)D_5 f_1] + 2D_{5,s}f_2\\
&\quad + D_{6,ss}(b^2 - s^2) + D_{6,s}(1 - n)s + 2nD_6\\
&\quad + \frac{1}{\widetilde{\alpha}^2}[D_{7,ss}\kappa_{00}(b^2 - s^2) - (3 + n)D_{7,s}\kappa_{00}s] + \frac{4}{\widetilde{\alpha}}(D_{7,s}f_1) + 2D_7\|\nabla\kappa\|^2,\\
\Sigma_2 &:= \widetilde{b}^i\widetilde{b}^j\mathbf{Ric}_{y^iy^j}\\
&= D_{1,ss}\|\nabla\kappa\|^2(b^2 - s^2)^2 + D_{1,s}\|\nabla\kappa\|^2 s(b^2 - s^2) + 2D_1\|\nabla\kappa\|^2 b^2\\
&\quad + \frac{1}{\widetilde{\alpha}^2}[D_{2,ss}\kappa_0^2(b^2 - s^2)^2 - 3D_{2,s}\kappa_0^2(b^2 - s^2)s] + \frac{4}{\widetilde{\alpha}}[D_{2,s}\kappa_0 f(b^2 - s^2)]\\
&\quad + 2D_2 f^2 + 2D_{3,s}f^2(b^2 - s^2) + 2D_3 f^2 s\\
&\quad + \frac{1}{\widetilde{\alpha}}[D_{3,ss}\kappa_0 f(b^2 - s^2)^2 - D_{3,s}\kappa_0 f(b^2 - s^2)s + D_3\kappa_0 f(b^2 - s^2)]\\
&\quad + D_{4,ss}f^2(b^2 - s^2)^2 + D_{4,s}f^2(b^2 - s^2)s + 2D_4 f^2 b^2\\
&\quad + \frac{1}{\widetilde{\alpha}}[D_{5,ss}f_1(b^2 - s^2)^2 - D_{5,s}f_1(b^2 - s^2)s + D_5 f_1(b^2 - s^2)]\\
&\quad + 2D_{5,s}f_2(b^2 - s^2) + 2D_5 f_2 s + D_{6,ss}(b^2 - s^2)^2 + D_{6,s}(b^2 - s^2)s\\
&\quad + 2D_6 b^2 + \frac{1}{\widetilde{\alpha}^2}[D_{7,ss}\kappa_{00}(b^2 - s^2)^2 - 3D_{7,s}\kappa_{00}(b^2 - s^2)s]\\
&\quad + \frac{4}{\widetilde{\alpha}}[D_{7,s}f_1(b^2 - s^2)] + 2D_7 f_2,\\
\Sigma_3 &:= (\widetilde{b}^i y^j + \widetilde{b}^j y^i)\mathbf{Ric}_{y^iy^j}\\
&= 2D_{1,s}\|\nabla\kappa\|^2(b^2 - s^2)\widetilde{\alpha} + 4D_1\|\nabla\kappa\|^2 s\widetilde{\alpha} + 4D_2\kappa_0 f
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{\widetilde{\alpha}}[D_{2,s}\kappa_0^2(b^2-s^2)]+2D_{3,s}\kappa_0 f(b^2-s^2)+2D_3 f(f\widetilde{\alpha}+\kappa_0 s)\\
&+2D_{4,s}f^2(b^2-s^2)\widetilde{\alpha}+4D_4 f^2 s\widetilde{\alpha}+2D_{5,s}f_1(b^2-s^2)+2D_5(f_2\widetilde{\alpha}+f_1 s)\\
&+2D_{6,s}(b^2-s^2)\widetilde{\alpha}+4D_6 s\widetilde{\alpha}+\frac{2}{\widetilde{\alpha}}[D_{7,s}\kappa_{00}(b^2-s^2)]+4D_7 f_1,
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_4 &:= y^i y^j \mathbf{Ric}_{y^i y^j}\\
&= 2D_1\|\nabla\kappa\|^2\widetilde{\alpha}^2+2D_2\kappa_0^2\\
&\quad+2D_3\kappa_0 f\widetilde{\alpha}+2D_4 f^2\widetilde{\alpha}^2+2D_5 f_1\widetilde{\alpha}+2D_6\widetilde{\alpha}^2+2D_7\kappa_{00}. \qquad \square
\end{aligned}
$$

Now we will prove Theorem 1.1. We take the famous coordinate transformation [16] to simplify the computations. Here we take an orthonnormal basis at x with respect to $\widetilde{\alpha}$ such that

$$
\widetilde{\alpha}=\sqrt{\sum_{i=1}^{n}(y^i)^2}, \qquad \widetilde{\beta}=\widetilde{b}y^1=by^1.
$$

Further, we take the following coordinate transformation [16] in T_xM, $\psi:(s,u^A)\to(y^i)$:

$$
y^1=\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha}, \qquad y^A=u^A,
$$

where $\overline{\alpha}=\sqrt{\sum_{i=2}^{n}(u^A)^2}$. Here, our index conventions are

$$
1\le i,j,k,\ldots\le n, \qquad 2\le A,B,C,\ldots\le n.
$$

We have

$$
\widetilde{\alpha}=\frac{b}{\sqrt{b^2-s^2}}\overline{\alpha}, \qquad \widetilde{\beta}=\frac{bs}{\sqrt{b^2-s^2}}\overline{\alpha}.
$$

Thus,

$$
F=e^{\kappa(x)}\widetilde{\alpha}\phi(s)=\frac{e^{\kappa(x)}b}{\sqrt{b^2-s^2}}\overline{\alpha}\phi(s), \quad s=\frac{\widetilde{\beta}}{\widetilde{\alpha}}.
$$

Further,

$$
\theta=\frac{t_1 s}{\sqrt{b^2-s^2}}\overline{\alpha}+\bar{t}_0,
$$

$$
f=\kappa_1 b, \quad f_1=\frac{bs\kappa_{11}}{\sqrt{b^2-s^2}}\overline{\alpha}+b\overline{\kappa}_{10}, \quad f_2=\kappa_{11}b^2,
$$

$$
\kappa_0=\frac{\kappa_1 s}{\sqrt{b^2-s^2}}\overline{\alpha}+\overline{\kappa}_0,\ \kappa_{00}=\frac{\kappa_{11}s^2}{b^2-s^2}\overline{\alpha}^2+\frac{2\overline{\kappa}_{10}s}{\sqrt{b^2-s^2}}\overline{\alpha}+\overline{\kappa}_{00},\ \|\nabla\kappa\|^2=\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2,
$$

where

$$
\bar{t}_0:=\sum_{A=2}^{n}t_A y^A, \qquad \overline{\kappa}_0:=\sum_{A=2}^{n}\kappa_A y^A,
$$

$$
\overline{\kappa}_{10}:=\sum_{A=2}^{n}\kappa_{1A}y^A, \qquad \overline{\kappa}_{00}:=\sum_{A,B=2}^{n}\kappa_{AB}y^A y^B.
$$

By this transformation, we can represent the equation $\mathbf{r} = n(n-1)[\frac{\theta}{F} + \mu]$ as

$$\mathbf{r} = n(n-1)\Big(\frac{t_1 s}{e^{\kappa} b\phi} + \frac{\bar{t}_0\sqrt{b^2-s^2}}{e^{\kappa} b\phi\overline{\alpha}} + \mu\Big), \tag{3.11}$$

and get the following lemma.

Lemma 3.4. *Let $F = e^{\kappa}\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is locally Minkowskian. Then the scalar curvature of F has the form $\mathbf{r} = n(n-1)(\frac{\theta}{F} + \mu)$ if and only if*

$$\Xi_2\overline{\alpha}^2 + \Xi_1\overline{\alpha} + \Xi_0 = 0, \tag{3.12}$$

where

$$\begin{aligned}
\Xi_2 := \frac{1}{b^2-s^2}\Big\{ & 2n(n-1)e^{\kappa}(t_1 bs + \mu e^{\kappa} b^2\phi)\overline{\alpha}^2(\phi - s\phi') \\
& - \Big\{\Big[D_{1,ss}(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)(b^2-s^2) + D_{1,s}(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)(1-n)s \\
& + 2nD_1(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2) + 2D_2(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2) + 2D_{3,s}b^2\kappa_1^2 \\
& + D_{4,ss}(b^2-s^2)b^2\kappa_1^2 + D_{4,s}(1-n)b^2 s\kappa_1^2 + 2nD_4 b^2\kappa_1^2 + 2D_{5,s}b^2\kappa_{11} \\
& + D_{6,ss}(b^2-s^2) + D_{6,s}(1-n)s + 2nD_6 + 2D_7(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)\Big]b^2 \\
& - (\eta\lambda^2 + \tau)\Big[D_{1,ss}(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)(b^2-s^2)^2 \\
& + D_{1,s}(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)s(b^2-s^2) + 2D_1(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2)b^2 \\
& + 2D_2 b^2\kappa_1^2 + 2D_{3,s}(b^2-s^2)b^2\kappa_1^2 + 2D_3 b^2 s\kappa_1^2 + D_{4,ss}(b^2-s^2)^2 b^2\kappa_1^2 \\
& + D_{4,s}(b^2-s^2)b^2 s\kappa_1^2 + 2D_4 b^4\kappa_1^2 + 2D_{5,s}b^2(b^2-s^2)\kappa_{11} + 2D_5 b^2 s\kappa_{11} \\
& + D_{6,ss}(b^2-s^2)^2 + D_{6,s}(b^2-s^2)s + 2D_6 b^2 + 2D_7 b^2\kappa_{11}\Big]b^2 \\
& - 2\eta\lambda\Big[D_{1,s}(b^2-s^2)(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2) + 2D_1 s(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2) + D_3 b^2\kappa_1^2 \\
& + D_{4,s}b^2(b^2-s^2)\kappa_1^2 + 2D_4 b^2 s\kappa_1^2 + D_5 b^2\kappa_{11} + D_{6,s}(b^2-s^2) + 2D_6 s\Big]b^2 \\
& - 2\eta\Big[D_1(\kappa_1^2 + \sum_{A=2}^{n}\kappa_A^2) + D_4 b^2\kappa_1^2 + D_6\Big]b^2 \\
& + \Big[4D_{2,s}bs\kappa_1^2 + D_{3,ss}(b^2-s^2)s\kappa_1^2 - (n+1)bs^2\kappa_1^2 + (n+1)D_3 bs\kappa_1^2 \\
& + D_{5,ss}(b^2-s^2)bs\kappa_{11} - (n+1)D_{5,s}bs^2\kappa_{11} + (n+1)D_5 bs\kappa_{11}
\end{aligned}$$

$$
\begin{aligned}
&+ 4D_{7,s}bs\kappa_{11}\Big]b - (\eta\lambda^2+\tau)\Big[4D_{2,s}(b^2-s^2)^2bs\kappa_1^2 + D_{3,ss}(b^2-s^2)^2bs\kappa_1^2\\
&- D_{3,s}(b^2-s^2)bs^2\kappa_1^2 + D_3bs(b^2-s^2)\kappa_1^2 + D_{5,ss}(b^2-s^2)^2bs\kappa_1^2\\
&- D_{3,s}(b^2-s^2)bs^2\kappa_1^2 + D_3bs(b^2-s^2)\kappa_1^2 + D_{5,ss}bs(b^2-s^2)^2\kappa_{11}\\
&- D_{5,s}bs^2(b^2-s^2)\kappa_{11} + D_5bs(b^2-s^2)\kappa_{11} + 4D_{7,s}bs(b^2-s^2)\kappa_{11}\Big]b\\
&- 2\eta\lambda\Big[2D_2bs\kappa_1^2 + D_{3,s}bs(b^2-s^2)\kappa_1^2 + D_3bs^2\kappa_1^2 + D_{5,s}bs(b^2-s^2)\kappa_{11}\\
&+ D_5bs^2\kappa_{11} + 2D_7bs\kappa_{11}]b - 2\eta[D_3bs\kappa_1^2 + D_5bs\kappa_{11}\Big]b\\
&+ D_{2,ss}s^2(b^2-s^2)\kappa_1^2 - (n+3)D_{2,s}s^3\kappa_1^2 + D_{7,ss}s^2(b^2-s^2)\kappa_{11}\\
&- (n+3)D_{7,s}s^3\kappa_{11} - (\eta\lambda^2+\tau)[D_{2,ss}s^2(b^2-s^2)^2\kappa_1^2\\
&- 3D_{2,s}(b^2-s^2)s^3\kappa_1^2 + D_{7,ss}s^2(b^2-s^2)^2\kappa_{11} - 3D_{7,s}s^3(b^2-s^2)\kappa_{11}]\\
&- 2\eta\lambda[D_{2,s}(b^2-s^2)s^2\kappa_1^2 + D_{7,s}(b^2-s^2)s^2\kappa_{11}]\\
&- 2\eta[D_2s^2\kappa_1^2 + D_7s^2\kappa_{11}]\Big\}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_1 :=\ & \frac{1}{\sqrt{b^2-s^2}}\Big\{2n(n-1)e^{\kappa}\bar{t}_0b(\phi - s\phi')\\
&- \Big\{ - 4\eta(\lambda b^2+s)D_2 + [-6(\eta\lambda^2+\tau)s^4 + 4\eta\lambda s^3\\
&+ (10b^2\eta\lambda^2 + 10b^2\tau - 2n - 6)s^2 - 4b^2\eta\lambda s - 4b^2(b^2\eta\lambda^2 + b^2\tau - 1)]D_{2,s}\\
&+ [(2(b^2-s^2)s - 2(\eta\lambda^2+\tau)(b^2-s^2)^2s)]D_{2,ss}\\
&+ [(n+1) - (\eta\lambda^2+\tau)(b^2-s^2) - 2\eta\lambda s - 2\eta]b^2D_3\\
&+ [-(n+1)s + (\eta\lambda^2)(b^2-s^2)s - 2\eta\lambda(b^2-s^2)]b^2D_{3,s}\\
&+ [(b^2-s^2) - (\eta\lambda^2+\tau)(b^2-s^2)^2]b^2D_{3,ss}\Big\}\kappa_1\overline{\kappa}_0\\
&- \Big\{[(n+1) - (\eta\lambda^2+\tau)(b^2-s^2) - 2\eta\lambda s - 2\eta]b^2D_5\\
&+ [-(n+1)s + (\eta\lambda^2)(b^2-s^2)s - 2\eta\lambda(b^2-s^2)]b^2D_{5,s}\\
&+ [(b^2-s^2) - (\eta\lambda^2+\tau)(b^2-s^2)^2]b^2D_{5,ss}\\
&- 4\eta(\lambda b^2+s)D_7 + [-6(\eta\lambda^2+\tau)s^4 + 4\eta\lambda s^3 + (10b^2\eta\lambda^2\\
&+ 10b^2\tau - 2n - 6)s^2 - 4b^2\eta\lambda s - 4b^2(b^2\eta\lambda^2 + b^2\tau - 1)]D_{7,s}\\
&+ [(2(b^2-s^2)s - 2(\eta\lambda^2+\tau)(b^2-s^2)^2s)]D_{7,ss}\Big\}\overline{\kappa}_{10}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_0 :=\ & -\Big\{D_{2,ss}(b^2-s^2)\overline{\kappa}_0^2 - D_{2,s}(n+3)s\overline{\kappa}_0^2 + D_{7,ss}(b^2-s^2)\overline{\kappa}_{00}\\
&- D_{7,s}(n+3)s\overline{\kappa}_{00} - (\eta\lambda^2+\tau)[D_{2,ss}(b^2-s^2)^2\overline{\kappa}_0^2 - 3D_{2,s}s(b^2-s^2)\overline{\kappa}_0^2\\
&+ D_{7,ss}(b^2-s^2)^2\overline{\kappa}_{00} - 3D_{7,s}s(b^2-s^2)\overline{\kappa}_{00}]
\end{aligned}
$$

$$- 2\eta\lambda[D_{2,s}(b^2 - s^2)\overline{\kappa}_0^2 + D_{7,s}(b^2 - s^2)\overline{\kappa}_{00}] - 2\eta[D_2\overline{\kappa}_0^2 + D_7\overline{\kappa}_{00}]\Big\}.$$

Since Ξ_1 is odd in u and Ξ_2, Ξ_0 are even in u, we can restate the above lemma as follows.

Lemma 3.5. *Let $F = e^\kappa \widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{\beta}/\widetilde{\alpha})$ is locally Minkowskian. Then the scalar curvature of F has the form $\mathbf{r} = n(n-1)(\frac{\theta}{F} + \mu)$ if and only if*

$$\Gamma_1 = \Gamma_2 = 0, \tag{3.13}$$

where

$$\begin{aligned}
\Gamma_1 := {} & 2n(n-1)e^\kappa t_A b(\phi - s\phi') \\
& - \Big\{ - 4\eta(\lambda b^2 + s)D_2 + [-6(\eta\lambda^2 + \tau)s^4 + 4\eta\lambda s^3 \\
& + (10b^2\eta\lambda^2 + 10b^2\tau - 2n - 6)s^2 - 4b^2\eta\lambda s - 4b^2(b^2\eta\lambda^2 + b^2\tau - 1)]D_{2,s} \\
& + [(2(b^2 - s^2)s - 2(\eta\lambda^2 + \tau)(b^2 - s^2)^2 s)]D_{2,ss} \\
& + [(n+1) - (\eta\lambda^2 + \tau)(b^2 - s^2) - 2\eta\lambda s - 2\eta]b^2 D_3 \\
& + [-(n+1)s + (\eta\lambda^2)(b^2 - s^2)s - 2\eta\lambda(b^2 - s^2)]b^2 D_{3,s} \\
& + [(b^2 - s^2) - (\eta\lambda^2 + \tau)(b^2 - s^2)^2]b^2 D_{3,ss}\Big\}\kappa_1\kappa_A \\
& - \Big\{[(n+1) - (\eta\lambda^2 + \tau)(b^2 - s^2) - 2\eta\lambda s - 2\eta]b^2 D_5 \\
& + [-(n+1)s + (\eta\lambda^2)(b^2 - s^2)s - 2\eta\lambda(b^2 - s^2)]b^2 D_{5,s} \\
& + [(b^2 - s^2) - (\eta\lambda^2 + \tau)(b^2 - s^2)^2]b^2 D_{5,ss} \\
& - 4\eta(\lambda b^2 + s)D_7 + [-6(\eta\lambda^2 + \tau)s^4 + 4\eta\lambda s^3 + (10b^2\eta\lambda^2 \\
& + 10b^2\tau - 2n - 6)s^2 - 4b^2\eta\lambda s - 4b^2(b^2\eta\lambda^2 + b^2\tau - 1)]D_{7,s} \\
& + [(2(b^2 - s^2)s - 2(\eta\lambda^2 + \tau)(b^2 - s^2)^2 s)]D_{7,ss}\Big\}\kappa_{1A}, \\
\Gamma_2 := {} & 2n(n-1)e^\kappa(t_1 bs + \mu e^\kappa b^2\phi)\overline{\alpha}^2(\phi - s\phi') \\
& - \Big\{\Big[D_{1,ss}(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2)(b^2 - s^2) + D_{1,s}(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2)(1-n)s \\
& + 2nD_1(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2) + 2D_2(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2) + 2D_{3,s}b^2\kappa_1^2 \\
& + D_{4,ss}(b^2 - s^2)b^2\kappa_1^2 + D_{4,s}(1-n)b^2 s\kappa_1^2 + 2nD_4 b^2\kappa_1^2 + 2D_{5,s}b^2\kappa_{11} \\
& + D_{6,ss}(b^2 - s^2) + D_{6,s}(1-n)s + 2nD_6 + 2D_7(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2)\Big]b^2 \\
& - (\eta\lambda^2 + \tau)\Big[D_{1,ss}(\kappa_1^2 + \sum_{A=2}^n \kappa_A^2)(b^2 - s^2)^2
\end{aligned}$$

$$
\begin{aligned}
&+ D_{1,s}(\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2)s(b^2-s^2)+2D_1(\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2)b^2\\
&+2D_2b^2\kappa_1^2+2D_{3,s}(b^2-s^2)b^2\kappa_1^2+2D_3b^2s\kappa_1^2+D_{4,ss}(b^2-s^2)^2b^2\kappa_1^2\\
&+D_{4,s}(b^2-s^2)b^2s\kappa_1^2+2D_4b^4\kappa_1^2+2D_{5,s}b^2(b^2-s^2)\kappa_{11}+2D_5b^2s\kappa_{11}\\
&+D_{6,ss}(b^2-s^2)^2+D_{6,s}(b^2-s^2)s+2D_6b^2+2D_7b^2\kappa_{11}\Big]b^2\\
&-2\eta\lambda\Big[D_{1,s}(b^2-s^2)(\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2)+2D_1s(\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2)+D_3b^2\kappa_1^2\\
&+D_{4,s}b^2(b^2-s^2)\kappa_1^2+2D_4b^2s\kappa_1^2+D_5b^2\kappa_{11}+D_{6,s}(b^2-s^2)+2D_6s\Big]b^2\\
&-2\eta\Big[D_1(\kappa_1^2+\sum_{A=2}^{n}\kappa_A^2)+D_4b^2\kappa_1^2+D_6\Big]b^2\\
&+\Big[4D_{2,s}bs\kappa_1^2+D_{3,ss}(b^2-s^2)s\kappa_1^2-(n+1)bs^2\kappa_1^2+(n+1)D_3bs\kappa_1^2\\
&+D_{5,ss}(b^2-s^2)bs\kappa_{11}-(n+1)D_{5,s}bs^2\kappa_{11}+(n+1)D_5bs\kappa_{11}\\
&+4D_{7,s}bs\kappa_{11}\Big]b-(\eta\lambda^2+\tau)\Big[4D_{2,s}(b^2-s^2)^2bs\kappa_1^2+D_{3,ss}(b^2-s^2)^2bs\kappa_1^2\\
&-D_{3,s}(b^2-s^2)bs^2\kappa_1^2+D_3bs(b^2-s^2)\kappa_1^2+D_{5,ss}(b^2-s^2)^2bs\kappa_1^2\\
&-D_{3,s}(b^2-s^2)bs^2\kappa_1^2+D_3bs(b^2-s^2)\kappa_1^2+D_{5,ss}bs(b^2-s^2)^2\kappa_{11}\\
&-D_{5,s}bs^2(b^2-s^2)\kappa_{11}+D_5bs(b^2-s^2)\kappa_{11}+4D_{7,s}bs(b^2-s^2)\kappa_{11}\Big]b\\
&-2\eta\lambda\Big[2D_2bs\kappa_1^2+D_{3,s}bs(b^2-s^2)\kappa_1^2+D_3bs^2\kappa_1^2+D_{5,s}bs(b^2-s^2)\kappa_{11}\\
&+D_5bs^2\kappa_{11}+2D_7bs\kappa_{11}]b-2\eta[D_3bs\kappa_1^2+D_5bs\kappa_{11}\Big]b\\
&+D_{2,ss}s^2(b^2-s^2)\kappa_1^2-(n+3)D_{2,s}s^3\kappa_1^2+D_{7,ss}s^2(b^2-s^2)\kappa_{11}\\
&-(n+3)D_{7,s}s^3\kappa_{11}-(\eta\lambda^2+\tau)[D_{2,ss}s^2(b^2-s^2)^2\kappa_1^2\\
&-3D_{2,s}(b^2-s^2)s^3\kappa_1^2+D_{7,ss}s^2(b^2-s^2)^2\kappa_{11}-3D_{7,s}s^3(b^2-s^2)\kappa_{11}]\\
&-2\eta\lambda[D_{2,s}(b^2-s^2)s^2\kappa_1^2+D_{7,s}(b^2-s^2)s^2\kappa_{11}]\\
&-2\eta[D_2s^2\kappa_1^2+D_7s^2\kappa_{11}]\Big\}\overline{\alpha}^2\\
&-\Big\{D_{2,ss}(b^2-s^2)^2\overline{\kappa}_0^2-D_{2,s}(n+3)s(b^2-s^2)\overline{\kappa}_0^2\\
&+D_{7,ss}(b^2-s^2)^2\overline{\kappa}_{00}-D_{7,s}(n+3)s(b^2-s^2)\overline{\kappa}_{00}\\
&-(\eta\lambda^2+\tau)[D_{2,ss}(b^2-s^2)^3\overline{\kappa}_0^2-3D_{2,s}s(b^2-s^2)^2\overline{\kappa}_0^2\\
&+D_{7,ss}(b^2-s^2)^3\overline{\kappa}_{00}-3D_{7,s}s(b^2-s^2)^2\overline{\kappa}_{00}]\\
&-2\eta\lambda[D_{2,s}(b^2-s^2)^2\overline{\kappa}_0^2+D_{7,s}(b^2-s^2)^2\overline{\kappa}_{00}]
\end{aligned}
$$

$$- 2\eta[D_2(b^2-s^2)\overline{\kappa}_0^2 + D_7(b^2-s^2)\overline{\kappa}_{00}]\Big\}.$$

Now we can prove Theorem 1.1 based on the above preparations.

Theorem 1.1. *Let $F = \alpha\phi(s), s = \beta/\alpha$ be a conformally flat (α, β)-metric on an n-dimensional manifold M with $\beta \neq 0$ and $n \geq 3$, where $\phi(s)$ is a polynomial of degree m $(m \geq 2)$. If F is of weakly isotropic scalar curvature $\mathbf{r}$, then $\mathbf{r} \equiv 0$.*

Proof. Assume that

$$\phi(s) = 1 + a_1 s + a_2 s^2 + \cdots + a_m s^m,\ \ a_m \neq 0.$$

Let

$$A_1 := \phi - s\phi', \qquad A_2 := \phi - s\phi' + (b^2 - s^2)\phi''.$$

Then we have

$$\eta = -\frac{((b^2-s^2)A_1 - b^2\phi)((b^2-s^2)A_1^2 - A_1 b^2\phi + A_2 s^2\phi)}{s^2\phi^2(b^2-s^2)A_2},$$

$$\eta\lambda = -\frac{(b^2-s^2)A_1^2 - A_1 b^2\phi + A_2 s^2\phi}{s\phi(b^2-s^2)A_2},$$

$$\eta\lambda^2 + \tau = \frac{A_2 - A_1}{A_2(b^2-s^2)}, \qquad \rho = A_1\phi.$$

By using the Maple program, multiplying $\Gamma_2 = 0$ by $4A_1^5A_2^7\phi^4 s^2$, we can express the result as a polynomial of s

$$E_{18m+2}s^{18m+2} + E_{18m+1}s^{18m+1} + \cdots + E_1 s + E_0 = 0, \tag{3.14}$$

where

$$E_{18m+2} := 8n(n-1)a_m^{18}(m-1)^{13}(m+1)^7\mu e^{2\kappa}b^2\overline{\alpha}^2.$$

From (3.14), we know that $E_i = 0$, $i = 0, 1, \ldots, 18m+2$. Because $m \geq 2$, $n \geq 3$, $a_m \neq 0$ and $b \neq 0$, we have $\mu(x) = 0$.

Taking $\mu(x) = 0$ into $\Gamma_2 = 0$. Similarly, we can get

$$E_{17m+3}s^{17m+3} + E_{17m+2}s^{17m+2} + \cdots + E_1 s + E_0 = 0, \tag{3.15}$$

where

$$E_{17m+3} := 8n(n-1)a_m^{17}(m-1)^{13}(m+1)^7 t_1 e^{\kappa} b\overline{\alpha}^2,$$

then we obtain $t_1 = 0$.

Next let us consider $\Gamma_1 = 0$. Similarly, it turns out

$$E_{17m+2}s^{17m+2} + E_{17m+1}s^{17m+1} + \cdots + E_1 s + E_0 = 0, \tag{3.16}$$

where

$$E_{17m+2} := 8n(n-1)a_m^{17}(m-1)^{13}(m+1)^7 t_A e^{\kappa} b\overline{\alpha},$$

so we have $t_A = 0$.

Hence, we obtain $\mathbf{r} = 0$ by $\mu = t_1 = t_A = 0$. □

Remark. When $m = 1$, F is a Randers metric. In this case, the method is no longer effected because the coefficient has the factor $(m-1)$. So whether Theorem 1.1 holds for Randers metric is true or not is not clear.

4. Proof of Theorem 1.2

Theorem 1.2. *Let $F=\alpha\phi(s), s=\beta/\alpha$ be a conformally flat (α,β)-metric on an n-dimensional manifold M with $n\geq 3$, where $\phi(s)=1+a_1s+a_2s^2+\cdots+a_{m-1}s^{m-1}+a_ms^m$ is a polynomial with $m\geq 2$ and $a_{m-1}a_m\neq 0$. If F is of weakly isotropic scalar curvature, then it must be locally Minkowskian or Riemannian.*

Proof. We follow Chen and Cheng's paper [3] and divide the proof into four steps.

Step 1. Proof of $\kappa_{AB}\kappa_{1C}=0$, $A\neq B$

Firstly let us consider $\Gamma_2=0$. Because of Theorem 1.1, $\mu=t_i=0$. It follows from $\Gamma_2=0$ that there is a function $\xi:=\xi(s)$ such that

$$p_2\kappa_A\kappa_B+p_7\kappa_{AB}=\xi(s)\delta_{AB}, \tag{4.1}$$

where

$$\begin{aligned} p_2=&-2\eta(b^2-s^2)D_2\\ &+[3(\eta\lambda^2+\tau)s(b^2-s^2)^2-2\eta\lambda(b^2-s^2)^2-(n+3)s(b^2-s^2)]D_{2,s}\\ &+[(b^2-s^2)^2-(\eta\lambda^2+\tau)(b^2-s^2)^3]D_{2,ss},\end{aligned}$$

and

$$\begin{aligned} p_7=&-2\eta(b^2-s^2)D_7\\ &+[3(\eta\lambda^2+\tau)s(b^2-s^2)^2-2\eta\lambda(b^2-s^2)^2-(n+3)s(b^2-s^2)]D_{7,s}\\ &+[(b^2-s^2)^2-(\eta\lambda^2+\tau)(b^2-s^2)^3]D_{7,ss}.\end{aligned}$$

Taking $A\neq B$ in (4.1) yields

$$p_2\kappa_A\kappa_B+p_7\kappa_{AB}=0. \tag{4.2}$$

Multiplying (4.2) by $4A_1^5A_2^7\phi^4s^2$ and by using the Maple program, we can get the following identity in s:

$$16(m-1)^{13}(m+1)^4m(n-2)[\kappa_A\kappa_B-(1+m)\kappa_{AB}]a_m^{16}s^{16m+4}+\sum_{j=0}^{16m+3}E_js^j=0.$$

It implies that $\kappa_A\kappa_B-(1+m)\kappa_{AB}=0(A\neq B)$. Plugging it into (4.2) yields

$$[(m+1)p_2+p_7]\kappa_{AB}=0,\quad A\neq B. \tag{4.3}$$

Now let us consider $\Gamma_1=0$. Multiplying $\Gamma_1=0$ by $4A_1^5A_2^7\phi^4s^2$ yields the following identity in s:

$$-32(m-1)^{13}(m+1)^4m(n-2)[\kappa_1\kappa_A-(1+m)\kappa_{1A}]a_m^{16}s^{16m+3}+\sum_{j=0}^{16m+2}E_js^j=0,$$

which implies that

$$\kappa_1\kappa_A-(1+m)\kappa_{1A}=0. \tag{4.4}$$

Plugging it back into $\Gamma_1=0$ yields

$$[(m+1)\frac{2s}{b^2-s^2}p_2+(m+1)(p_2{}'+p_3)+\frac{2s}{b^2-s^2}p_7+(p_7{}'+p_5)]\kappa_{1A}=0. \tag{4.5}$$

Where

$$\begin{aligned}p_2{}'&=-4\eta\lambda b^2D_2+4b^2[1-(\eta\lambda^2+\tau)(b^2-s^2)]D_{2,s},\\ p_7{}'&=-4\eta\lambda b^2D_7+4b^2[1-(\eta\lambda^2+\tau)(b^2-s^2)]D_{7,s},\end{aligned}$$

and

$$\begin{aligned}p_3=&[(n+1)-(\eta\lambda^2+\tau)(b^2-s^2)-2\eta\lambda s-2\eta]b^2D_3\\ &+[-(n+1)s+(\eta\lambda^2)(b^2-s^2)s-2\eta\lambda(b^2-s^2)]b^2D_{3,s}\\ &+[(b^2-s^2)-(\eta\lambda^2+\tau)(b^2-s^2)^2]b^2D_{3,ss},\\ p_5=&[(n+1)-(\eta\lambda^2+\tau)(b^2-s^2)-2\eta\lambda s-2\eta]b^2D_5\\ &+[-(n+1)s+(\eta\lambda^2)(b^2-s^2)s-2\eta\lambda(b^2-s^2)]b^2D_{5,s}\\ &+[(b^2-s^2)-(\eta\lambda^2+\tau)(b^2-s^2)^2]b^2D_{5,ss}.\end{aligned}$$

Multiplying (4.5) by $\kappa_{AB}(A\neq B)$ and by (4.3), we have

$$[(m+1)(p_2{}'+p_3)+(p_7{}'+p_5)]\kappa_{AB}\kappa_{1C}=0. \tag{4.6}$$

Multiplying (4.6) by $4A_1^5A_2^7\phi^4s^2(b^2-s^2)$, we obtain

$$E_{16m+3}s^{16m+3}+E_{16m+2}s^{16m+2}+\sum_{j=0}^{16m+1}E_js^j=0,$$

where

$$E_{16m+3}=-2(m-1)^{10}(m+1)^5mT(m,n)a_m^{16}b^2\kappa_{AB}\kappa_{1C},$$

and

$$\begin{aligned}T(m,n)=&-20m^4n^2+16m^5+530m^4n+18m^3n^2-1220m^4-1105m^3n\\ &+14m^2n^2+3534m^3+567m^2n-18mn^2-2557m^2-27mn\\ &+364m+6n^2-29n-9.\end{aligned}$$

If $T(m,n)\neq 0$, one can easily get $\kappa_{AB}\kappa_{1C}=0, A\neq B$.

Now we prove $T(m,n)\neq 0$, if $T(m,n)=0$, we can get two roots of n by

$$n_1=A(m)+B(m),$$
$$n_2=A(m)-B(m),$$

where

$$A(m):=\frac{530m^4-1105m^3+567m^2-27m-29}{4(10m^4-9m^3-7m^2+9m-3)},$$
$$B(m):=\frac{\sqrt{C(m)}}{4(10m^4-9m^3-7m^2+9m-3)},$$

and

$$C(m) := 1280m^9 + 182148m^8 - 801636m^7 + 1432509m^6 - 1354594m^5 + 750411m^4 - 255184m^3 + 55923m^2 - 7818m + 1057.$$

Firstly we consider n_2. We have $n_2 < 2$ since $m \geq 2$, thus $n \neq n_2$.

Next we consider n_1.

(i) For $m < 100$, n_1 can not be an integer by a direct computation.

(ii) For $m \geq 100$, we have

$$A(m) \leq \frac{53}{4} < 14, \qquad B(m) \leq \sqrt{2m}.$$

Thus

$$n = n_1 < 14 + \sqrt{2m}. \tag{4.7}$$

On the other hand, $T(m, n) = 0$ implies

$$m \mid 6n^2 - 29n - 9.$$

For $n = 3, 4, 5$, $T(m, n) = 0$ has no integer solution for m.

For $n \geq 6$, we have $6n^2 - 29n - 9 > 0$. We can write

$$6n^2 - 29n - 9 = mq, \qquad q \geq 1.$$

Take $m = \frac{6n^2-29n-9}{q}$ into $T(m, n) = 0$. For $q < 48$, $T(m, n) = 0$ has no integer solution for n by numerical calculation.

For $q \geq 48$, we have

$$6n^2 \geq mq \geq 48m.$$

Thus

$$n \geq 2\sqrt{2m}. \tag{4.8}$$

Compare with (4.7) and (4.8) we have

$$2\sqrt{2m} < 14 + \sqrt{2m}.$$

It is a contradiction for $m \geq 100$, thus $n \neq n_1$.

Hence we obtain $T(m, n) \neq 0$ and

$$\kappa_{AB}\kappa_{1C} = 0, \quad A \neq B.$$

Step 2. Proof of $\boldsymbol{\kappa_{1A} \equiv 0}$.

We claim that $\kappa_{1A} \equiv 0$ on M. Suppose that there exist an integer A_0 ($2 \leq A_0 \leq n$) and a point $x_0 \in M$ such that $\kappa_{1A_0}(x_0) \neq 0$, then there must be a neighborhood $U_{x_0} \subset M$ of x_0 such that $\kappa_{1A_0}(x) \neq 0$, $\forall x \in U_{x_0}$. Because of (4.4) we know $\kappa_1\kappa_{A_0} = (1 + m)\kappa_{1A_0}$, we get $\kappa_{A_0}(x) \neq 0$, $\forall x \in U_{x_0}$. By Step 1 we know $\kappa_{AB}\kappa_{1A_0} = 0$ ($A \neq B$), then we have $\kappa_{AB}(x) = 0$ ($A \neq B$), $\forall x \in U_{x_0}$. Noting that $\kappa_{A_0}\kappa_B = (1 + m)\kappa_{A_0B}$ ($A_0 \neq B$), we obtain $\kappa_B(x) = 0$, $\forall x \in U_{x_0}$ and $B \neq A_0$. Then we have $\kappa_{BB}(x) = 0$, $\forall x \in U_{x_0}$ and $B \neq A_0$.

Taking $A = B \neq A_0$ in (4.1) at the point $x_0 \in M$ yields

$$\xi(s) = p_2\kappa_B^2(x_0) + p_7\kappa_{BB}(x_0) = 0.$$

Then (4.2) holds for all A and B which implies that $\kappa_A\kappa_B - (1+m)\kappa_{AB} = 0$ for all A and B. It means the analysis in Step 1 is also true for $A = B$. In particular, we have $\kappa^2_{A_0}(x_0) = (m+1)\kappa_{A_0A_0}(x_0)$. Then we can get $[(m+1)p_2 + p_7]\kappa_{A_0A_0}(x_0) = 0$. Note that $\kappa_{A_0A_0}(x_0) = [\frac{\kappa^2_{A_0}(x_0)}{m+1}] \neq 0$, we conclude that $(m+1)p_2 + p_7 = 0$. Plugging it into the (4.5) and by $\kappa_{1A_0}(x_0) \neq 0$, we obtain

$$(m+1)({p_2}' + p_3) + ({p_7}' + p_5) = 0. \tag{4.9}$$

From the analysis in Step 1 we know (4.9) is not true. Thus we have $\kappa_{1A} = 0$.

Step 3. Proof of $\boldsymbol{\kappa_A \equiv 0}$.

Suppose that there exist an integer B_0 $(2 \leq B_0 \leq n)$ and a point $x_1 \in M$ such that $\kappa_{B_0}(x_1) \neq 0$, then there is a neighborhood $U_{x_1} \subset M$ of x_1 such that $\kappa_{B_0}(x) \neq 0$, $\forall x \in U_{x_1}$. Note that $\kappa_1\kappa_{B_0} = (m+1)\kappa_{1B_0} = 0$, we have $\kappa_1(x) = \kappa_{11}(x) = 0$, $\forall x \in U_{x_1}$. Then the $\Gamma_2 = 0$ can be reduced to

$$(p_1 \sum_{A=2}^{n} \kappa^2_A + p_{61} \sum_{A=2}^{n} \kappa_{AA})\overline{\alpha}^2 + p_2\overline{\kappa}^2_0 + p_7\overline{\kappa}_{00} = 0, \tag{4.10}$$

which implies that there is a function $\gamma = \gamma(s)$ such that

$$p_2\kappa_A\kappa_B + p_7\kappa_{AB} = \gamma(s)\delta_{AB}. \tag{4.11}$$

By (4.10) we have

$$((p_1 \sum_{A=2}^{n} \kappa^2_A + p_{61} \sum_{A=2}^{n} \kappa_{AA}) + p_2\kappa^2_B + p_7\kappa_{BB} = 0. \tag{4.12}$$

Further, taking sum of B we have

$$[(n-1)p_1 + p_2] \sum_{A=2}^{n} \kappa^2_A + [(n-1)p_{61} + p_7] \sum_{A=2}^{n} \kappa_{AA} = 0.$$

Multiplying it by $4A_1^5A_2^6\phi^2$, we get

$$\begin{aligned} &- 16(m-1)^{13}(m+1)^4 m(n-2)a_m^{16}[\sum_{A=2}^{n} \kappa^2_A - (m+1) \sum_{A=2}^{n} \kappa_{AA}]s^{16m+6} \\ &+ \sum_{j=0}^{16m+5} E_j s^j = 0. \end{aligned}$$

The above equation implies that

$$\sum_{A=2}^{n} \kappa^2_A = (m+1) \sum_{A=2}^{n} \kappa_{AA}. \tag{4.13}$$

It follows from (4.11) that

$$p_2(\kappa^2_A - \kappa^2_B) + p_7(\kappa_{AA} - \kappa_{BB}) = 0. \tag{4.14}$$

Plugging the expressions of p_2 and p_7 into above equation, we can obtain the following identity in s,

$$E_{16m+4}s^{16m+4}+\sum_{j=0}^{16m+3}E_js^j=0,$$

where

$$E_{16m+4}=16(m-1)^{13}(m+1)^4m(n-2)[\kappa_A^2-\kappa_B^2-(1+m)(\kappa_{AA}-\kappa_{BB})]a_m^{16}.$$

It implies that $\kappa_A^2-\kappa_B^2=(1+m)(\kappa_{AA}-\kappa_{BB})$. Then we have

$$(n-1)\kappa_A^2-\sum_{B=2}^n\kappa_B^2=(n-1)(m+1)\kappa_{AA}-(m+1)\sum_{B=2}^n\kappa_{BB}.$$

Thus by (4.13) we obtain

$$\kappa_A^2=(m+1)\kappa_{AA}.$$

Plugging it into (4.12) yields

$$([(m+1)p_1+p_{61}]\sum_{A=2}^n\kappa_{AA}+[(m+1)p_2+p_7]\kappa_{BB}=0. \tag{4.15}$$

(i) If $(m+1)p_2+p_7\neq 0$, we have $\kappa_{AA}=\kappa_{BB}$ from (4.14), and then, $\kappa_A=\pm\kappa_B$. Note that $\kappa_A\kappa_B-(m+1)\kappa_{AB}=0(A\neq B)$. Taking $A\neq B$ in (4.11), we have

$$\begin{aligned}0&=p_2\kappa_A\kappa_B+p_7\kappa_{AB}=p_2\kappa_A\kappa_B+p_7[(\kappa_A\kappa_B)/(m+1)]\\&=\pm[(m+1)p_2+p_7]\kappa_A^2(x)/(m+1).\end{aligned}$$

Taking $x=x_1$ and $A=B_0$, we have $\pm[(m+1)p_2+p_7]\kappa_{B_0}^2(x_1)/(m+1)=0$. This is a contradiction.

(ii) If $(m+1)p_2+p_7=0$, (4.15) reduced to

$$[(m+1)p_1+p_{61}]\sum_{A=2}^n\kappa_{AA}=0.$$

The assumption that $\kappa_{B_0}(x_1)\neq 0$ implies that

$$\sum_{A=2}^n\kappa_{AA}=\sum_{A=2}^n\kappa_A^2/(m+1)\neq 0$$

at the point $x_1\in M$. Thus we have

$$(m+1)p_1+p_{61}=0.$$

Multiplying the equation above by $4A_1^5A_2^6\phi^2$, we can obtain

$$E_{16m+4}s^{16m+4}+E_{16m+3}s^{16m+3}+\sum_{j=0}^{16m+2}E_js^j=0,$$

where

$$E_{16m+4} = 8(m-1)^{10}(m+1)^6 H(m,n) b^2 a_m^{16},$$

and

$$H(m,n) := (-m^4 + m^3 - m + 1)n^2 + (2m^5 - 4m^3 - 3m^2 + 2m - 1)n - 4m^5 + 10m^4 + 4m^3 - 2m^2.$$

If $H(m,n) = 0$, we have two solutions $n = n_1$ and $n = n_2$ by

$$n_1 = \frac{2m^5 - 4m^3 - 3m^2 + 2m - 1 + Q(m)}{2(m^4 - m^3 + m - 1)},$$

and

$$n_2 = \frac{2m^5 - 4m^3 - 3m^2 + 2m - 1 - Q(m)}{2(m^4 - m^3 + m - 1)},$$

where

$$Q(m) := \sqrt{4m^{10} - 16m^9 + 40m^8 - 36m^7 - 16m^6 + 84m^5 - 31m^4 - 28m^3 + 18m^2 - 4m + 1}.$$

For n_2, one can easily obtain $n_2 < 2$ since $m \geq 2$.

For n_1, we have

$$0 < n_1 - 2m < 1.$$

It means that n_1 is not an integer. Then we obtain $H(m,n) \neq 0$, thus (4.15) is not true.

Hence, we have $\kappa_A(x) = 0$, $\forall x \in M$.

Step 4. Proof of $\kappa_1 \equiv 0$

Finally, plugging $\kappa_A = 0$ into $\Gamma_2 = 0$ and multiplying it by $4A_1^5 A_2^6 \phi^2$, by using the Maple program, we can get

$$E_{16m+6} s^{16m+6} + E_{16m+5} s^{16m+5} + \sum_{j=0}^{16m+4} E_j s^j = 0, \tag{4.16}$$

where

$$E_{16m+6} = 16(m-1)^{13}(m+1)^4 m(n-2)[\kappa_1^2 - (1+m)\kappa_{11}] a_m^{16}.$$

Then we obtain $\kappa_1^2 = (1+m)\kappa_{11}$. Then the E_{16m+5} reduced to

$$E_{16m+5} = -24(m-1)^{11}(m+1)^4 m a_m^{15} a_{m-1}[(n-2)m - n + 3] b^2 \kappa_{11}.$$

Because of $(n-2)m - n + 3 \neq 0$ and $a_{m-1} \neq 0$, we obtain that $\kappa_1 = \kappa_{11} = 0$.

Hence $\kappa_i = 0$, that is, $\kappa(x)$ is a constant. Thus F is locally Minkowskian. □

In the end we will consider a special case $\phi(s) = 1 + a_m s^m$.

Proof of Proposition 1.4. By the assumption that $\phi(s) = 1 + a_m s^m$. If $a_m = 0$, it is a Riemann metric. Thus we always assume that $a_m \neq 0$ in the following. By a series similarly analysis, we can get $\kappa_A \equiv 0$.

In this special case, (4.16) becomes

$$E_{16m+6}s^{16m+6}+E_{16m+4}s^{16m+4}+\sum_{j=0}^{16m+3}E_js^j=0,$$

where

$$E_{16m+6}=16(m-1)^{13}(m+1)^4m(n-2)[\kappa_1^2-(1+m)\kappa_{11}]a_m^{16}.$$

Then we obtain $\kappa_1^2=(1+m)\kappa_{11}$. Now the E_{16m+4} becomes

$$E_{16m+4}=64(m-1)^{10}(m+1)^5W(m,n)a_m^{16}b^2\kappa_{11},$$

where

$$\begin{aligned}W(m,n)=&\ m^2(m^3-m^2+2)b^2+\frac{3}{8}(m-1)(m^4+\frac{4}{3}m^3-\frac{2}{3}m^2+\frac{1}{3})n^2\\&+\frac{3}{4}(m-1)(m^5+\frac{31}{12}m^4+\frac{1}{2}m^3-\frac{23}{12}m^2-\frac{1}{3}m+\frac{1}{6})n\\&+m^2(-\frac{15}{8}m^4+\frac{3}{8}m^3+\frac{33}{8}m^2-\frac{15}{8}m-\frac{3}{4}).\end{aligned}$$

By a direct computation we have $W(m,n)>\frac{3}{8}m^6+\frac{615}{4}>0$ with $m\geq 2$, $n\geq 3$, thus we obtain $\kappa_1=\kappa_{11}=0$.

Hence $\kappa_i=0$, that is, $\kappa(x)$ is a constant. Thus F is locally Minkowskian. □

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